

Ehlers-Harrison transformations and black holes in dilaton-axion gravity with multiple vector fields

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Dilaton-axion gravity with p $U(1)$ vector fields is studied on space-times admitting a timelike Killing vector field. A three-dimensional σ model is derived in terms of Kähler geometry, and a holomorphic representation of the $SO(2,2+p)$ global symmetry is constructed. A general static black hole solution depending on $2p+5$ parameters is obtained via $SO(2,2+p)$ covariantization of the Schwarzschild solution. The metric in the curvature coordinates looks like the variable mass Reissner-Nordström one and generically possesses two horizons. The inner horizon is pushed to the singularity if electric and magnetic $SO(p)$ charge vectors are parallel. For nonparallel charges the inner horizon has a finite area except for an extremal limit when this property is preserved only for orthogonal charges. Extremal dyon configurations with orthogonal charges have finite horizon radii continuously varying from zero to the Arnowitt-Deser-Misner mass. The new general solution is endowed with a Newman-Unti-Tamburino parameter, asymptotic values of dilaton and axion, and a gauge parameter which can be used to ascribe any given value to one of the scalar charges. [S0556-2821(97)07304-9]

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I. INTRODUCTION

Recently there has been substantial progress in understanding the statistical origin of black hole entropy in string theory (for a review and references see [1]). In achieving this goal it was important to realize that black holes in theories with dilaton and multiple vector fields may have Bogomol'ni-Prasad-Sommerfield (BPS) saturated states with the finite area of the horizon. For toroidally compactified heterotic string the effective four-dimensional theory is $N=4$ supergravity coupled to Abelian vector multiplets. The above feature is manifest already at the level of the pure $N=4$ supergravity. The bosonic sector of this theory consists of the gravity coupled system of six $U(1)$ vector fields interacting with a dilaton and axion. It is well known that, with only one $U(1)$ component excited, the dilaton causes the horizon of the extremal black hole to shrink to a singularity [2–4]. With more than one $U(1)$ charge present, some BPS configurations may possess a finite horizon area such as the Reissner-Nordström solution. This was demonstrated by Kallosh *et al.* [5] within the two-vectors model without an axion. Extremal black holes with a finite horizon area have $N=1$ residual supersymmetry [5,6] contrary to an extremal “dilaton” black hole with a vanishing horizon area, which exhibits two supersymmetries unbroken [7]. Similar conclusions were made in the heterotic string effective theory with additional vector multiplets present [8].

In view of the above it is important to investigate the space of classical solutions to pure and matter coupled $N=4$, $D=4$ supergravity in more detail. Although many

particular black hole solutions were obtained earlier using various solution generating techniques, or solving Bogomol'nyi-type equations in the BPS limit [9], some recent study [6] revealed that this theory still requires further analysis. Of particular importance are stationary solutions which can be explored using powerful integration methods developed earlier in the vacuum general relativity and Einstein-Maxwell theory [10]. Their generalization to dilaton-axion gravity interacting with one vector field was elaborated recently [11,12]. The approach consists in the derivation of the three-dimensional σ model, revealing the isometries of the target manifold (potential space), and utilizing its Kähler structure to achieve a more concise formulation. This procedure has direct similarity with the Ernst approach to General Relativity [13] based on the introduction of complex potentials parameterizing Kähler target manifolds $SL(2,R)/SO(2)$ (vacuum) and $SU(2,1)/[SU(2) \times U(1)]$ (electrovacuum) [14]. It brings considerable simplifications into solution generating techniques both at the level of the finite-dimensional symmetry group, and the infinite-dimensional Geroch-type symmetries if further assumption of the second spacetime symmetry (axial) is made [15,16].

The Kähler parametrization of the target manifold of the stationary dilaton-axion gravity with multiple vector fields was given in [17]. The purpose of the present paper is to develop the corresponding solution generating technique and construct the most general nonrotating black hole solution to $N=4$ theory in a form manifestly covariant under the three-dimensional U -duality group. We classify isometries of the target manifold in usual terms of general relativity [Ehlers-Harrison transformations, electric-magnetic duality, gauge, scale transformations, and $SO(p)$ rotations], and give explicitly the corresponding holomorphic maps [p being the number of $U(1)$ vector fields]. Choosing particular combinations which preserve asymptotic conditions, we apply them to the Schwarzschild solution to covariantize it with respect to the

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U -duality group $SO(2,2+p)$. Thus a new solution depending on $2p+5$ real parameters is generated.

Our solution includes many particular previously known black hole configurations, and allows one to determine their position in the general parameter space. The metric depends on five parameters [including the Newman-Unti-Tamburino (NUT) parameter], while the material fields are determined by two charge vectors, asymptotic values of the dilaton and axion, and a gauge parameter which can be used to generate solutions with a prescribed value of one of scalar charges. The metric is presented in the curvature coordinates facilitating its physical interpretation and analysis of the BPS limit. It is shown that *generic nonextremal* black hole possesses two nonsingular horizons and has a modified Reissner-Nordström form unless charges are parallel. If they are parallel, the internal horizon shrinks to the singularity and the metric reduces to that of the “dilaton” black hole [2,3]. In the case of orthogonal charges of equal norm the metric is pure Reissner-Nordström (thus we prove that this configuration found previously in [5] is unique). In the general case the metric deviates from the usual Reissner-Nordström one by a “variable mass” term, asymptotically equal to Arnowitt-Deser-Misner (ADM) mass and depending on one additional parameter. In the BPS limit the (without a NUT parameter) solution with nonorthogonal charges has a “dilaton” form (event horizon shrinking to the singularity). When charges are orthogonal but have nonequal norms, the radius of the event horizon in curvature coordinates varies between zero value (for the nondyon case) and the ADM mass (for the case of equal norms).

The paper is organized as follows. In Sec. II the four-dimensional theory is reviewed and the symmetries of the equations of motions are briefly discussed. Then we perform $1+3$ decomposition of the metric and derive a σ -model representation in three dimensions (Sec. III). Infinitesimal isometries of the target manifold are presented in Sec. IV. The complex parameterization of the target manifold is introduced in Sec. V, where the holomorphic maps corresponding to the target space isometries are constructed. Section VI is devoted to the covariantization of the Schwarzschild solution with respect to U -duality of the three-dimensional theory. In Sec. VII we discuss the choice of coordinates, analyze various particular black hole configurations, and study the BPS limit. We conclude with brief remarks clarifying the relationship to previously done works.

II. PRELIMINARIES

We consider the gravity coupled system on p Abelian vector fields A_μ^a , $a=1, \dots, p$, one scalar ϕ (dilaton), and one pseudoscalar κ (axion) described by the action

$$S = \frac{1}{16\pi} \int \left\{ -R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu}^a F^{a\mu\nu} - \kappa F_{\mu\nu}^a \tilde{F}^{a\mu\nu} \right\} \sqrt{-g} d^4x, \quad (2.1)$$

where $\tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\tau} F_{\lambda\tau}^a$, $F^a = dA^a$, R is the (four-dimensional) scalar curvature corresponding to metric $g_{\mu\nu}$,

sum over repeated a is understood. For $p=6$ this is the bosonic part of the $N=4, D=4$ supergravity.

Equations for vector fields following from this action have a form of modified Maxwell equations:

$$\nabla_\nu (e^{-2\phi} F^{a\mu\nu} + \kappa \tilde{F}^{a\mu\nu}) = 0, \quad (2.2)$$

together with the Bianchi identities

$$\nabla_\nu \tilde{F}^{a\mu\nu} = 0. \quad (2.3)$$

The dilaton satisfies the curved space D'Alembert equation with axion and vector sources:

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{2} e^{-2\phi} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} e^{4\phi} (\nabla \kappa)^2, \quad (2.4)$$

while the axion is generated by the pseudoscalar invariant of vector fields

$$\nabla^\mu (e^{4\phi} \nabla_\mu \kappa) = -F_{\mu\nu}^a \tilde{F}^{a\mu\nu}. \quad (2.5)$$

The system is closed by the Einstein equations with vector and scalar sources

$$R_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} e^{4\phi} \nabla_\mu \kappa \nabla_\nu \kappa + e^{-2\phi} \left(2F_{\mu\lambda}^a F^{a\lambda}{}_\nu + \frac{1}{2} F_{\lambda\tau}^a F^{a\lambda\tau} g_{\mu\nu} \right), \quad (2.6)$$

where $(\nabla \kappa)^2 \equiv g^{\mu\nu} (\nabla_\mu \kappa) (\nabla_\nu \kappa)$, and ∇_μ is the covariant derivative with respect to the four-dimensional metric $g_{\mu\nu}$.

The action (2.1) is invariant under the global $SO(p)$ rotations of vector fields:

$$A_\mu^a \rightarrow \Omega^a{}_b A_\mu^b, \quad \Omega^a{}_b \Omega^b{}_c = \delta_c^a. \quad (2.7)$$

Another symmetry of the equations of motion, known as S -duality [18], can be concisely expressed using the complex axidilaton variable

$$z = \kappa + i e^{-2\phi}. \quad (2.8)$$

In terms of z the action (2.1) can be rewritten as

$$S = -\frac{1}{16\pi} \int \left\{ R + 2\nabla_z \nabla_{\bar{z}} (z - \bar{z})^{-2} - 2\text{Re}(iz \mathcal{F}_{\mu\nu}^a \mathcal{F}^{a\mu\nu}) \right\} \sqrt{-g} d^4x, \quad (2.9)$$

where $\mathcal{F}^a = (F^a + i\tilde{F}^a)/2$, $\tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\tau} F_{\lambda\tau}^a$ (bar denotes complex conjugation). Then the (four-dimensional) S -duality transformations read

$$z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \mathcal{F}_{\mu\nu}^a \rightarrow (\gamma \bar{z} + \delta) \mathcal{F}^{a\mu\nu}, \quad \alpha \delta - \beta \gamma = 1. \quad (2.10)$$

They leave invariant the kinetic term of the z -field in the action (2.9) as well as the full set of equations of motion (but not the action).

It is worth comparing the present theory with the Einstein-Maxwell one (bosonic sector of $N=2$ supergravity):

$$S = -\frac{1}{16\pi} \int \{R + 2(\mathcal{F}_{\mu\nu}^a \bar{\mathcal{F}}^{a\mu\nu})\} \sqrt{-g} d^4x. \quad (2.11)$$

In this case both the Maxwell equations and Bianchi identities can be combined into one complex equation

$$\nabla_\mu \mathcal{F}^{a\mu\nu} = 0, \quad (2.12)$$

while the Einstein equations read

$$R_{\mu\nu} = 4\mathcal{F}_{(\mu\lambda}^a \bar{\mathcal{F}}_{\nu)}^{a\lambda}. \quad (2.13)$$

The electric-magnetic duality now is an Abelian symmetry of the full action:

$$\mathcal{F}_{\mu\lambda}^a \rightarrow e^{i\alpha} \mathcal{F}_{\mu\lambda}^a, \quad (2.14)$$

where α is a constant parameter. Notice that the action of the dilaton-axion theory (2.9), which has two additional field variables with respect to Eq. (2.11), possesses at the same time larger symmetries [three-parametric $SL(2, R)$ instead of one-parametric $U(1)$]. Because of this enhancement of four-dimensional symmetries, further three-dimensional reduction of the dilaton-axion gravity is describable as a nonlinear σ model on a symmetric space similarly to the Einstein-Maxwell theory.

III. 3+1 DECOMPOSITION

Let us consider the system of equations (2.2)–(2.6) on a class of metrics depending effectively on three coordinates. In a coordinate-independent way this can be expressed as a restriction on spacetime to admit a Killing vector field. It turns out that, if the Killing field is non-null everywhere, the full system of equations can be presented in the form of a gravity coupled nonlinear σ model on the symmetric space $SO(2, 2+p)/[SO(2) \times SO(2+p)]$ or one of its noncompact forms. Aiming to investigate black holes, we assume here that the Killing vector is timelike in an essential region of spacetime. Then ten components of $g_{\mu\nu}$ can be expressed using the standard Kaluza-Klein ansatz through six components of the three-space metric h_{ij} , one (co)vector ω_i , ($i, j = 1, 2, 3$), and one scalar f , depending on the space coordinates x^i :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(dt - \omega_i dx^i)^2 - \frac{1}{f} h_{ij} dx^i dx^j. \quad (3.1)$$

To decompose vector fields, one has first to introduce the set of electric potentials $A_0^a = v^a/\sqrt{2}$ (the coefficient helps to avoid undesired numerical factors in the three-dimensional formulation). The spatial parts of four potentials can be traded for another set of scalars, magnetic potentials u^a , using Eqs. (2.2) for $\nu = i$. Since spatial part of the tensor at the left-hand side is divergenceless, one can write

$$e^{-2\phi} F^a{}_{ij} + \kappa \tilde{F}^a{}_{ij} = \frac{f}{\sqrt{2}h} \epsilon^{ijk} \partial_k u^a. \quad (3.2)$$

This relation can be put into an alternative form

$$\tilde{F}_{0i}^a = \frac{e^{2\phi}}{\sqrt{2}} w_k^a, \quad w_k^a = v(\partial_k u^a - \kappa \partial_k v^a), \quad (3.3)$$

and w_k^a can be considered as covariant components of the three-dimensional vector \mathbf{w}^a .

The remaining Maxwell equations and Bianchi identities now lead to the following set of three-dimensional equations for v^a, u^a :

$$\nabla(f^{-1} e^{-2\phi} \nabla v^a) + f^{-2} \boldsymbol{\tau} \nabla u^a - \nabla(f^{-1} \kappa e^{2\phi} \mathbf{w}^a) = 0, \quad (3.4)$$

$$\nabla(f^{-1} e^{2\phi} \mathbf{w}^a) - f^{-2} \boldsymbol{\tau} \nabla v^a = 0, \quad (3.5)$$

where the three-dimensional vector $\boldsymbol{\tau}$ dual to two-form $d\omega$ is introduced

$$\tau^i = -f^2 \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_j \omega_k. \quad (3.6)$$

Here (and in what follows) Latin indices are raised and lowered using the three-metric h_{ij} and its inverse h^{ij} , and ∇ denotes a three-space covariant derivative.

To clarify the role of $\boldsymbol{\tau}$, the mixed components of the Ricci tensor should be invoked [19]:

$$R_0^i = -\frac{f}{2\sqrt{h}} \epsilon^{ijk} \partial_k \tau_j. \quad (3.7)$$

Upon substitution in the source of the corresponding components of the Einstein equations, one finds

$$\epsilon^{ijk} \partial_k \tau_j = 2\epsilon^{ijk} \partial_j v^a \partial_k u^a. \quad (3.8)$$

This equation can be integrated by introducing the NUT (twist) potential χ :

$$\tau_i = \partial_i \chi + v^a \partial_i u^a - u^a \partial_i v^a. \quad (3.9)$$

Multiplying Eq. (3.9) by f^{-2} , and taking into account Eq. (3.6), one obtains the following second-order equation for the NUT potential:

$$f(\Delta \chi + v^a \Delta u^a - u^a \Delta v^a) = 2(\nabla \chi + v^a \nabla u^a - u^a \nabla v^a) \nabla f, \quad (3.10)$$

where

$$\Delta = \nabla^2 = \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j) \quad (3.11)$$

is a three-dimensional Laplacian. From the R_{00} component of the Einstein equations one obtains an equation for the metric function $f = g_{00}$ in a similar form:

$$f \Delta f - (\nabla f)^2 + \boldsymbol{\tau}^2 = f[e^{2\phi} (\mathbf{w}^a)^2 + e^{-2\phi} (\nabla v^a)^2]. \quad (3.12)$$

Finally, the dilaton and axion equations (2.4) and (2.5) reduce to

$$\Delta\phi = \frac{1}{2f}[e^{-2\phi}(\nabla v^a)^2 - e^{2\phi}(\mathbf{w}^a)^2] + \frac{e^{4\phi}}{2}(\nabla\kappa)^2, \quad (3.13)$$

$$\nabla(e^{4\phi}\nabla\kappa) = \frac{2}{f}e^{2\phi}\mathbf{w}^a\nabla v^a. \quad (3.14)$$

Now one can check that this system of equations is derivable from the following three-dimensional action:

$$S_m = \int \left\{ \frac{1}{2f^2}[(\nabla f)^2 + (\nabla\chi + v^a\nabla u^a - u^a\nabla v^a)^2] 2(\nabla\phi)^2 + \frac{1}{2}e^{4\phi}(\nabla\kappa)^2 - \frac{1}{f}[e^{2\phi}(\mathbf{w}^a)^2 + e^{-2\phi}(\nabla v^a)^2] \right\} \sqrt{h}d^3x, \quad (3.15)$$

which describes a nonlinear σ model with $2p+4$ scalar fields $\varphi^A = (f, \chi, v^a, u^a, \kappa, \phi)$, $A=1, \dots, 2p+4$. The remaining (spatial) Einstein equations reduce to three-dimensional Einstein equations for the metric h_{ij} :

$$\begin{aligned} \mathcal{R}_{ij} = & \frac{1}{2f^2}(f_{,i}f_{,j} + \tau_i\tau_j) + 2\phi_{,i}\phi_{,j} + \frac{1}{2}e^{4\phi}\kappa_{,i}\kappa_{,j} \\ & - \frac{1}{f}(e^{-2\phi}v_{,i}^av_{,j}^a + e^{2\phi}w_i^aw_j^a), \end{aligned} \quad (3.16)$$

where \mathcal{R}_{ij} is the three-dimensional Ricci tensor. Comparing with Eq. (3.15), one can see that the source term is derivable from the same action (3.15) as the energy-momentum tensor. Therefore, the full system of equations follows from the three-dimensional gravity coupled σ -model action

$$S_\sigma = \int (-\mathcal{R} + \mathcal{G}_{AB}\partial_i\varphi^A\partial_j\varphi^B h^{ij})\sqrt{h}d^3x, \quad (3.17)$$

where the target space metric can directly read off from Eq. (3.15) as follows:

$$dl^2 = \frac{1}{2f^2}\{df^2 + (d\chi + v^adu^a - u^adv^a)^2\} + 2d\phi^2 + \frac{1}{2}e^{4\phi}d\kappa^2 - \frac{1}{f}\{e^{2\phi}(du^a - \kappa dv^a)^2 + e^{-2\phi}(dv^a)^2\}. \quad (3.18)$$

The target manifold is a $(2p+4)$ -dimensional pseudo-Euclidean space of the signature $2p-4$ which encodes the hidden global symmetries of the stationary dilaton-axion gravity with p vector fields ($p=1$ theory was studied earlier in [11]). In absence of scalar fields it reduces (up to generalization to arbitrary p) to the potential space of Neugebauer and Kramer for the stationary Einstein-Maxwell system [20]. It is worth noting, however, that the target space of the Einstein-Maxwell theory does not constitute a subspace of the full target manifold (3.18). Contrary to this, the *vacuum* Einstein target manifold, parametrized by f, χ is a subspace. This means that there is an intrinsic connection between stationary solutions of the vacuum (but not electrovacuum) General Relativity and solutions to the present theory.

IV. U-DUALITY

Global symmetries of the three-dimensional theory are manifest as isometries of the target space. As far as we have an explicit expression for the target space metric, we can explore its isometries by solving Killing equations

$$K_{A;B} + K_{B;A} = 0, \quad (4.1)$$

where covariant derivatives refer to the metric (3.18). A general solution to Eq. (4.1) was given in [21] for a particular case $p=1$. Generalization to arbitrary p is rather straightforward in the gauge and S -duality sectors, but is nontrivial in the Ehlers-Harrison sector. Here we give the full set of solutions to Eq. (4.1) for arbitrary p .

A. Gauge transformations

This sector includes $2p+1$ linear constant shifts of electric and magnetic potentials and a NUT potential. No physical quantities are changed. Since electric and magnetic variables are mixed with a NUT potential in Eq. (3.9), the electromagnetic gauge transformations include appropriate variations of χ

$$v^a \rightarrow v^a + e^a, \quad \chi \rightarrow \chi - u^a e^a, \quad (4.2)$$

$$u^a \rightarrow u^a + m^a, \quad \chi \rightarrow \chi + v^a m^a, \quad (4.3)$$

$$\chi \rightarrow \chi + g, \quad (4.4)$$

where e^a, m^a , and g are real parameters. The corresponding Killing vectors are

$$K_a^e = \partial_{v^a} - u^a \partial_\chi, \quad K_a^m = \partial_{u^a} + v^a \partial_\chi, \quad K^g = \partial_\chi. \quad (4.5)$$

B. Scale and SO(p) rotations

An invariance of Eq. (3.18) under rescaling

$$f \rightarrow e^{2s}f, \quad \chi \rightarrow e^{2s}\chi, \quad v^a \rightarrow e^s v^a, \quad u^a \rightarrow e^s u^a \quad (4.6)$$

is obvious, with κ and ϕ unchanged. This transformation changes the four-dimensional metric. The corresponding Killing vector is

$$K^s = 2f\partial_f + 2\chi\partial_\chi + v^a\partial_{v^a} + u^a\partial_{u^a}. \quad (4.7)$$

Rotations in the space of U(1) potentials (2.7) induces the corresponding transformations of electric and magnetic potentials:

$$v^a \rightarrow \Omega^a_b v^b, \quad u^a \rightarrow \Omega^a_b u^b. \quad (4.8)$$

They are generated by

$$K_{ab} = v^a\partial_{v^b} - v^b\partial_{v^a} + u^a\partial_{u^b} - u^b\partial_{u^a}. \quad (4.9)$$

C. S duality

Four-dimensional S duality (2.10) induces the corresponding transformations of the target space variables. They consist of the axion shift

$$\kappa \rightarrow \kappa + d_1, \quad u^a \rightarrow u^a + d_1 v^a, \quad (4.10)$$

proper electric-magnetic rotations

$$u^a \rightarrow u^a, \quad v^a \rightarrow v^a + u^a d_2, \quad z^{-1} \rightarrow z^{-1} + d_2, \quad (4.11)$$

and $SL(2, R)$ scale transformation

$$z \rightarrow e^{-2d_3} z, \quad u^a \rightarrow e^{-d_3} u^a, \quad v^a \rightarrow e^{d_3} v^a. \quad (4.12)$$

The corresponding Killing generators

$$\begin{aligned} K^{d_1} &= \partial_\kappa + v^a \partial_{u^a}, \quad K^{d_2} = (e^{-4\phi} - \kappa^2) \partial_\kappa + \kappa \partial_\phi + u^a \partial_{v^a}, \\ K^{d_3} &= \partial_\phi - 2\kappa \partial_\kappa + v^a \partial_{v^a} - u^a \partial_{u^a} \end{aligned} \quad (4.13)$$

satisfy $SL(2, R)$ commutation relations

$$\begin{aligned} [K^{d_3}, K^{d_1}] &= 2K^{d_1}, \quad [K^{d_3}, K^{d_2}] = -2K^{d_2}, \\ [K^{d_1}, K^{d_2}] &= K^{d_3}. \end{aligned} \quad (4.14)$$

D. Ehlers-Harrison sector

The nontrivial part of the isometry group includes $2p$ Harrison transformations whose action on real target space variables is rather involved (for the $p=1$ case see [11]). Fortunately, more concise form exists in terms of complex variables, which we discuss in the next section. Here we consider only infinitesimal transformations. Note, that they are not direct generalizations of those for $p=1$ [11]. The first set (electric) is

$$\begin{aligned} K_a^{H_1} &= 2v^a f \partial_f + v^a \partial_\phi + 2w^a \partial_\kappa + (fe^{2\phi} - v^2) \partial_{v^a} \\ &+ 2v^b (v^a \partial_{v^b} + u^a \partial_{u^b}) + (\chi - u^b v^b + \kappa fe^{2\phi}) \partial_{u^a} \\ &+ [v^a (\chi + v^b w^b) + w^a (fe^{2\phi} - v^2)] \partial_\chi, \end{aligned} \quad (4.15)$$

where $v^2 = (v^a)^2$, and we do not distinguish between upper and lower $SO(p)$ indices. The main effect of these transformations is to generate electric potentials when acting on electrically uncharged configurations. Their magnetic counterpart reads

$$\begin{aligned} K_a^{H_2} &= 2u^a f \partial_f + (\kappa v^a - w^a) \partial_\phi + 2(\kappa w^a + v^a e^{-4\phi}) \partial_\kappa \\ &+ 2u^b (v^a \partial_{v^b} + u^a \partial_{u^b}) - (u^b v^b + \chi - \kappa fe^{2\phi}) \partial_{v^a} \\ &+ (fe^{-2\phi} - u^2 + \kappa^2 fe^{2\phi}) \partial_{u^a} \\ &+ [u^a \chi + v^a (u^b w^b - fe^{-2\phi}) \\ &+ w^a (\kappa fe^{2\phi} - v^b u^b)] \partial_\chi, \end{aligned} \quad (4.16)$$

where $w^a = u^a - \kappa v^a$, $u^2 = (u^a)^2$.

The last, Ehlers-type [22] generator, which is produced through the commutation relations

$$[K_a^{H_1}, K_a^{H_2}] = 2\delta_{ab} K^E, \quad (4.17)$$

reads

$$\begin{aligned} K^E &= 2f\chi \partial_f + w^b v^b \partial_\phi + (w^2 - v^2 e^{-4\phi}) \partial_\kappa + [v^b (\chi + v^a w^a) \\ &+ w^b (fe^{2\phi} - v^2)] \partial_{v^b} + [u^b \chi + v^b (w^2 + \kappa v^a w^a - fe^{-2\phi}) \\ &+ w^b (\kappa fe^{2\phi} - \kappa v^2 - v^a w^a)] \partial_{u^b} + [\chi^2 - f^2 + f v^2 e^{-2\phi} \\ &+ w^2 (fe^{2\phi} - v^2) + (v^a w^a)^2] \partial_\chi, \end{aligned} \quad (4.18)$$

where $w^2 = (w^a)^2$. Altogether these Killing vectors form a closed $[(p+3)(p+4)/2]$ -dimensional algebra. To identify it with $SO(2, p+2)$ let us consider a $(4+p)$ -dimensional space endowed with the pseudo-Euclidean metric

$$d\sigma^2 = G_{\mu\nu} d\xi^\mu d\xi^\nu = -(d\xi^0)^2 - (d\xi^\theta)^2 + (d\xi^1)^2 + \dots + (d\xi^{p+2})^2, \quad (4.19)$$

and denote the corresponding $SO(2, p+2)$ generators as $L_{\mu\nu}$, where $\mu, \nu = 0, \theta, 1, \dots, p+2$ and $a, b = 1, \dots, p$. Then the following correspondence between $L_{\mu\nu}$ and the above Killing vectors can be established:

$$L_{ab} = K_{ab}, \quad L_{0a} = \frac{1}{2}(K_a^{H_2} - K_a^m), \quad L_{0p+1} = \frac{1}{2}(K^{d_3} - K^s),$$

$$L_{0p+2} = \frac{1}{2}(K^g - K^{d_1} - K^{d_2} - K^E), \quad L_{ap+1} = \frac{1}{2}(K_a^{H_2} + K_a^m),$$

$$L_{p+1p+2} = \frac{1}{2}(K^g - K^{d_1} + K^{d_2} + K^E),$$

$$L_{ap+2} = \frac{1}{2}(K_a^{H_1} + K_a^e),$$

$$L_{p+1\theta} = \frac{1}{2}(K^g + K^{d_1} + K^{d_2} - K^E),$$

$$L_{\theta p+2} = -\frac{1}{2}(K^{d_3} + K^s),$$

$$L_{0\theta} = \frac{1}{2}(K^g + K^{d_1} - K^{d_2} + K^E), \quad L_{\theta a} = \frac{1}{2}(K_a^{H_1} - K_a^e). \quad (4.20)$$

The target space (3.18) can now be identified with the coset manifold $SO(2, p+2)/[SO(2) \times SO(2, p)]$.

V. HOLOMORPHIC REPRESENTATION

The coset manifold $SO(2, p+2)/[SO(2) \times SO(2, p)]$ may be parametrized by $p+2$ complex coordinates and endowed with the Kählerian metric [17]. These coordinates can be regarded as generalization of Ernst potentials of the Einstein-Maxwell theory. The set consists of the axidilaton z , p complex electromagnetic potentials

$$\Phi^a = u^a - z v^a \quad (5.1)$$

and the gravitational complex potential

$$E = if - \chi + v^a \Phi^a. \quad (5.2)$$

In absence of vector fields Eq. (5.2) reduces to the original Ernst potential $\epsilon = f + i\chi$ multiplied by i . However, when there are no scalar fields, E does not reduce to the Ernst potential of the Einstein-Maxwell theory. In particular, the E -potential is not symmetric under an interchange $v^a \leftrightarrow u^a$ contrary to its Einstein-Maxwell counterpart. This reflects an intrinsic asymmetry of the $N=4$ supergravity with respect to electric and magnetic sectors.

In terms of complex variables the target space metric (3.18) can be rewritten as

$$dl^2 = 1/(2V^2) |\text{Im}z| dE + 2\text{Im}\Phi^a d\Phi^a - (\text{Im}\Phi^a)^2 (dz/\text{Im}z)|^2 - (1/V) |d\Phi^a - \text{Im}\Phi^a (dz/\text{Im}z)|^2 + \frac{1}{2} |dz/\text{Im}z|^2, \quad (5.3)$$

where the quantity

$$V = \text{Im}E \text{Im}z + (\text{Im}\Phi^a)^2 \quad (5.4)$$

is related to the Kähler potential K as [17]

$$V = e^{-K}. \quad (5.5)$$

The Kählerian metric (5.3) is generated by K via

$$dl^2 = 2(\partial_\alpha \bar{\partial}_{\bar{\beta}} K) dz^\alpha d\bar{z}^\beta, \quad (5.6)$$

where holomorphic variables z^α are enumerated as

$$z^\alpha = (E, \Phi^a, z), \quad \alpha = 0, 1, \dots, p+1. \quad (5.7)$$

Now the isometries of the target space may be presented as holomorphic maps $z^\alpha \rightarrow z'^\alpha(z^\beta)$ leaving K invariant up to an admissible transformation:

$$K \rightarrow K + h(z) + \bar{h}(\bar{z}), \quad (5.8)$$

where h is an arbitrary holomorphic function of the Kähler coordinates. First we describe two discrete maps which simplify substantially the derivation of holomorphic counterparts to continuous isometry transformations. One of them consists in the interchange of the Ernst potential E and the axidilaton:

$$E \rightarrow z, \quad z \rightarrow E, \quad \Phi^a \text{ unchanged}. \quad (5.9)$$

This transformation does not modify V (5.4) and hence leave the target space metric unchanged. In terms of the real variables it corresponds to

$$f \rightarrow \frac{f}{fe^{2\phi} - v^2}, \quad e^{2\phi} \rightarrow f - v^2 e^{2\phi}, \quad v^a \rightarrow \frac{v^a}{fe^{2\phi} - v^2},$$

$$w^a v^a - \chi \leftrightarrow \kappa, \quad (5.10)$$

where $v^2 = (v^a)^2$, while u remains unchanged. The second discrete transformation is more sophisticated. It corresponds to an interchange of the Ernst and axidilaton variables accompanied by the coordinate-dependent rescaling

$$E \rightarrow h^{-1} z, \quad z \rightarrow h^{-1} E, \quad \Phi^a \rightarrow h^{-1} \Phi^a, \quad (5.11)$$

where

$$h(z^\alpha) = Ez + \Phi^2, \quad \Phi^2 \equiv (\Phi^a)^2 \quad (5.12)$$

is the function entering into the corresponding (admissible) transformation of the Kähler potential (5.8). This transformation is rather complicated being expressed in terms of real variables, but Kähler representation makes it easy to check. This map has another interpretation in the case $p=1$ [12,17] as an inversion of the “matrix Ernst potential.”

Now we are in a position to list holomorphic maps corresponding to continuous isometries of the target space. The two simplest maps, linear shifts of E and Φ^a on real constants g and m^a , correspond to gravitational

$$E' = E - g, \quad \Phi'^a = \Phi^a, \quad z' = z \quad (5.13)$$

and magnetic

$$E' = E, \quad \Phi'^a = \Phi^a + m^a, \quad z' = z \quad (5.14)$$

gauge transformation (4.3) and (4.4). An electric gauge transformation looks somewhat more complicated,

$$z' = z, \quad \Phi'^a = \Phi^a - e^a z, \quad E' = E + 2e^a \Phi^a - (e^a)^2 z, \quad (5.15)$$

while the scale transformation (4.6) is simply rewritten as

$$E' = e^{2s} E, \quad \Phi'^a = e^s \Phi^a, \quad z' = z. \quad (5.16)$$

Now apply the discrete map (5.9) to the electric gauge transformation (4.2). Decomposing real Killing vectors of the previous section into holomorphic and antiholomorphic parts one obtains

$$K_a^e = -z \partial_{\Phi^a} + 2\Phi^a \partial_E, \quad (5.17)$$

so that after Eq. (5.9) we get

$$K_a^{H1} = 2\Phi^a \partial_z - E \partial_{\Phi^a}, \quad (5.18)$$

which can be identified with the holomorphic part of the electric Harrison Killing vector (4.15). This means that the finite electric Harrison holomorphic map may be constructed via an interchange of E and z in Eq. (5.15):

$$E' = E, \quad \Phi^a = \Phi^a + \nu^a E, \quad z' = z - 2\nu^a \Phi^a - \nu^2 E, \quad (5.19)$$

where ν^a is a set of real parameters, and $\nu^2 = (\nu^a)^2$.

Similarly, applying the discrete map (5.11) to the magnetic gauge (4.3), which infinitesimally reads

$$K_a^m = \partial_{\Phi^a}, \quad (5.20)$$

one obtains

$$K_a^{H2} = (Ez + \Phi^2) \partial_{\Phi^a} - 2\Phi^a (z \partial_z + E \partial_E) - 2\Phi^a \Phi^b \partial_{\Phi^c}. \quad (5.21)$$

This can be identified with the holomorphic part of the magnetic Harrison Killing vector (4.16), and thus applying Eq. (5.11) to the finite magnetic gauge transformation (5.14) one obtains the holomorphic map corresponding to finite magnetic Harrison transformations

$$E' = \frac{E}{\Lambda}, \quad \Phi'^a = \frac{\Phi^a + \mu^a}{\Lambda}, \quad z' = \frac{z}{\Lambda},$$

$$\Lambda = 1 + 2\mu^a \Phi^a + \mu^2 (Ez + \Phi^2), \quad (5.22)$$

where $\mu^2 = (\mu^a)^2$. Applying Eq. (5.11) on the gravitational gauge in the same way

$$K_g = -\partial_E, \quad (5.23)$$

one obtains Ehlers holomorphic generator

$$K^E = E(\partial_E + \Phi^a \partial_{\Phi^a}) - \Phi^2 \partial_z. \quad (5.24)$$

The corresponding finite form is

$$E' = \frac{E}{1 + cE}, \quad \Phi'^a = \frac{\Phi^a}{1 + cE}, \quad z' = z + \frac{c\Phi^2}{1 + cE}. \quad (5.25)$$

The remaining S -duality transformations can be obtained in a similar way by applying Eq. (5.9) to the gravitational gauge (5.13), scale (5.16), and Ehlers transformations. This results in

$$E' = E, \quad \Phi'^a = \Phi^a, \quad z' = z + d_1, \quad (5.26a)$$

$$E' = E + \frac{d_2 \Phi^2}{1 + d_2 z}, \quad \Phi' = \frac{\Phi^a}{1 + d_2 z}, \quad z' = \frac{z}{1 + d_2 z}, \quad (5.26b)$$

$$E' = E, \quad \Phi' = e^{d_3} \Phi^a, \quad z' = e^{2d_3} z. \quad (5.26c)$$

Thus we have constructed U -duality finite $SO(2,2+p)$ transformations in terms of holomorphic maps acting on complex potentials which can be regarded as Ernst-type variables. In the case $p=1$ our holomorphic maps coincide with those given previously in [12], where they were found in a different way using a symplectic representation of the $SO(2,3)$ algebra. In the next section we will consider their application to solution generation purposes.

VI. $SO(2,2+p)$ COVARIANTIZATION OF THE SCHWARZSCHILD SOLUTION

The target space of the vacuum Einstein gravity $SL(2, \mathbb{R})/SO(2)$ can be parametrized by a single complex variable $E_0 = if - \chi$ and form a subspace of Eq. (5.3). Using holomorphic transformations of the previous section one can construct a fully $SO(2,2+p)$ covariant counterpart to any vacuum solution of the Einstein equations. An application of this procedure to the Schwarzschild metric is likely to produce a general static black hole solution to dilaton-axion gravity with multiple vector fields. Brief results were reported recently [23]. Here we give a more detailed derivation and discuss further properties of the solution obtained.

In terms of holomorphic potentials the Schwarzschild solution reads

$$E_0 = if_0, \quad f_0 = 1 - \frac{2M_0}{r_0}, \quad z_0 = i, \quad \Phi_0^a = 0, \quad (6.1)$$

and the three metric (which remains nonaffected by transformations) has nonzero components $h_{rr} = 1$, $h_{\theta\theta} = f_0 r_0^2$, $h_{\varphi\varphi} = f_0 r_0^2 \sin^2 \theta$. Let us first formulate the asymptotic conditions for holomorphic variables ensuring asymptotically flat (Taub-NUT) behavior of the solution. It is worth noting that the NUT parameter enters as one of the $SO(2,2+p)$ charges, so it has to be included in order to maintain the $SO(2,2+p)$ covariance.

If asymptotic values of the dilaton and axion are allowed to take arbitrary values $\phi(\infty) = \phi_\infty$, $\kappa(\infty) = \kappa_\infty$, the charges should be defined as follows in the limit $r \rightarrow \infty$:

$$E \sim i \left(1 - \frac{2\mathcal{M}}{r} \right), \quad z \sim z_\infty - \frac{2i\mathcal{D}}{r} e^{-2\phi_\infty},$$

$$\Phi^a \sim -\frac{i\sqrt{2}\mathcal{Q}^a}{r} e^{-\phi_\infty}, \quad (6.2)$$

where $z_\infty = \kappa_\infty + ie^{-2\phi_\infty}$ and three complex charges are introduced $\mathcal{M} = M + iN$ (ADM mass and NUT parameter), $\mathcal{D} = D + iA$ (dilaton and axion), $\mathcal{Q}^a = Q^a + iP^a$ (electric and magnetic). Holomorphic transformations listed in the previous section can be combined into several sets preserving the asymptotic conditions (6.2) [23]. It is convenient to perform first the charging (Harrison) transformations and then Ehlers and one of the S -duality rotations imposing a condition $z_\infty = i$, and afterwards restore an arbitrary value of z_∞ via Eq. (5.26). So, as an initial step, we apply Harrison transformations (first magnetic and then electric) combining them with suitable magnetic and electric gauge and axidilaton rescaling to preserve conditions $E(\infty) = z(\infty) = i$, $\Phi(\infty) = 0$. This results in

$$E = \frac{im\tilde{n}_f + 2q(1-f_0)}{nm_f}, \quad z = \frac{imn_f - 2q(1-f_0)}{nm_f},$$

$$\Phi^a = \frac{(1-f_0)[\mu^a n + \nu^a(2q-im)]}{nm_f}, \quad (6.3)$$

where

$$q = \nu^a \mu^a, \quad n = 1 - \nu^2, \quad m = 1 - \mu^2,$$

$$m_f = 1 - \mu^2 f_0, \quad n_f = 1 - \nu^2 f_0, \quad \tilde{m}_f = f_0 - \mu^2,$$

$$\tilde{n}_f = f_0 - \nu^2. \quad (6.4)$$

From here one can extract the following expressions for the metric function f and the NUT potential:

$$f = \frac{mnf_0}{m_f n_f}, \quad \chi = \frac{q(f_0^2 - 1)}{m_f n_f}. \quad (6.5)$$

The latter has a nonvanishing Coulomb term in the asymptotic expansion, and hence, according to Eq. (6.2), the solutions are asymptotically Taub-NUT with $N = 2qM_0/mn$. [One can check that this is the only effect produced by χ ; short-range terms in χ are exactly compensated by the contribution of vector fields in Eq. (3.9), so the metric does not contain any other terms than NUT parameters due to $\omega_i dx^i$.]

The drawback of this solution is that the NUT parameter is not free but is determined by charge parameters, and it is nonzero unless the $SO(p)$ vectors μ^a , ν^a are orthogonal, or one of them is zero. A NUT parameter of the solution, however, can be made arbitrary by applying the Ehlers transformation (5.25), what can be easily seen considering its action on the asymptotics (6.2). Similar role in the S -duality subgroup is played by the map (5.26b): it rotates dilaton and axion charges in the same way as Ehlers map rotates the ADM mass and NUT. Combining both these rotations with gauge and scale transformations to preserve asymptotic conditions [23], one obtains for the complex variable E the following expression:

$$E = \frac{2q(1-f_0)(1+bc) - n(cm_f + b\tilde{m}_f) + im(\tilde{n}_f - bc\tilde{n}_f)}{2q(1-f_0)(c-b) + n(m_f - bc\tilde{m}_f) + im(bn_f + c\tilde{n}_f)}, \quad (6.6)$$

where c is the parameter of the Ehlers transformation, and $b=d_1$ is that of (5.26a). The corresponding values of the ADM mass and NUT parameter depend only on c :

$$M = \frac{M_0}{mn(1+c^2)} \{(1-\mu^2\nu^2)(1-c^2) + 4cq\},$$

$$N = \frac{2M_0}{mn(1+c^2)} \{c(\mu^2\nu^2 - 1) + (1-c^2)q\}. \quad (6.7)$$

Choosing c to be the root (regular at $q=0$) of the equation

$$q(c^2 - 1) + c(1 - \mu^2\nu^2) = 0, \quad (6.8)$$

one selects strictly asymptotically flat solutions without a NUT parameter. In this case

$$M = \frac{M_0(1+c^2)}{mn(1-c^2)} (1 - \mu^2\nu^2), \quad (6.9)$$

where

$$c = -\frac{1 - \mu^2\nu^2}{2q} + \{1 + [(1 - \mu^2\nu^2)/2q]^2\}^{1/2}. \quad (6.10)$$

If $M_0 \geq 0$, the ADM mass of the new solution is non-negative if $\mu^2 \leq 1$, $\nu^2 \leq 1$, $c^2 \leq 1$ (the limiting values should be considered more carefully in view of the singularity). These restrictions on the parameters will be assumed so forth. Alternatively, by an appropriate choice of c one can construct massless solutions; these are nontrivial ($N \neq 0$).

Finally, arbitrary asymptotic values of dilaton and axion can be generated via Eqs. (5.26a) and (5.26c), the metric being unaffected. The resulting complex axidilaton field reads

$$z = \kappa_\infty - e^{-2\phi_\infty} \frac{2q(1-f_0)(1+bc) + n(c\tilde{m}_f + bm_f) - im(n_f - bc\tilde{n}_f)}{2q(1-f_0)(c-b) + n(m_f - bc\tilde{m}_f) + im(bn_f + c\tilde{n}_f)}. \quad (6.11)$$

According to asymptotics (6.2) one obtains the following values of dilaton and axion charges in terms of the parameters introduced:

$$D = \frac{M_0}{mn(1+b^2)} \{(\mu^2 - \nu^2)(1-b^2) - 4bq\},$$

$$A = \frac{2M_0}{mn(1+b^2)} \{b(\nu^2 - \mu^2) - (1-b^2)q\}. \quad (6.12)$$

Hence b plays for D and A the same role as c for ADM mass and NUT charge. In particular, any of scalar charges D , A may be set zero by an appropriate choice of b .

The remaining set of $U(1)$ complex potentials is given by

$$\Phi^a = \frac{e^{-\phi_\infty}(1+c^2)^{1/2}(1+b^2)^{1/2}(1-f_0)\{\mu^a n + \nu^a(2q-im)\}}{2q(1-f_0)(c-b) + n(m_f - bc\tilde{m}_f) + im(bn_f + c\tilde{n}_f)}, \quad (6.13)$$

while the corresponding electric and magnetic charges are

$$Q^a = \frac{\sqrt{2}M_0\{(n\mu^a + 2q\nu^a)(b+c) + m\nu^a(1-bc)\}}{mn(1+b^2)^{1/2}(1+c^2)^{1/2}},$$

$$P^a = \frac{\sqrt{2}M_0\{(n\mu^a + 2q\nu^a)(1-bc) - m\nu^a(b+c)\}}{mn(1+b^2)^{1/2}(1+c^2)^{1/2}}. \quad (6.14)$$

Altogether the solution contains $2p+5$ arbitrary parameters M_0 , μ^a , ν^a , b , c , κ_∞ , ϕ_∞ . However, the number of independent physical charges together with asymptotic values of dilaton and axion is $2p+4$. Indeed, both dilaton and axion charges [as can be directly checked using Eqs. (6.7), (6.12), and (6.14)] are determined by the equations

$$D = \frac{M\{(P^a)^2 - (Q^a)^2\} - 2NQ^aP^a}{2(M^2 + N^2)},$$

$$A = \frac{N\{(Q^a)^2 - (P^a)^2\} - 2MQ^aP^a}{2(M^2 + N^2)}, \quad (6.15)$$

through electric and magnetic charges, mass and NUT parameter, or, in the complex form,

$$\mathcal{D} = -\frac{(Q^a)^2}{2\mathcal{M}}, \quad (6.16)$$

and thus are not independent parameters. Meanwhile the uniqueness of the theorem holds [24] for harmonic maps to symmetric spaces (which is the present case too) saying that the solution is uniquely determined by the Coulomb charges. Moreover, the metric depends, in fact, only on three combinations of ν^a, μ^a ; namely $\mu^2, \nu^2, q = \mu^a \nu^a$ [the only scalar invariants which can be built from $SO(p)$ vectors]. Also it can be shown that the metric is b independent. Extracting the function f from Eqs. (6.6), (6.11), and (6.13), one finds

$$f = \frac{f_0 m n (1 + c^2)}{m_f n_f + c^2 \tilde{n}_f \tilde{m}_f + 2c q (1 - f_0^2)}, \quad (6.17)$$

that is, the metric is determined by ν^2, μ^2, q, c, M_0 and a NUT parameter entering through χ . All this indicates that b is a gauge parameter. It is, however, useful to keep it arbitrary if one wishes to get solutions with a prescribed value of one of the scalar charges. In particular, for

$$b = \frac{2q}{\nu^2 - \mu^2} + \frac{\mu^2 - \nu^2}{|\mu^2 - \nu^2|} \{1 + [4q^2/(\nu^2 - \mu^2)]^2\} \quad (6.18)$$

[as in Eq. (6.10) we choose the root which remains regular for all values of the parameters] the dilaton charge will be zero.

The real potentials v^a, u^a, ϕ, κ take simpler forms in a “symmetric” gauge $b = c$. Then the scalar fields are

$$e^{-2(\phi - \phi_\infty)} = \frac{m\{m_f n_f + c^2 \tilde{n}_f \tilde{m}_f + 2c q (1 - f_0^2)\}}{n(m_f^2 + c^2 \tilde{m}_f^2)}, \quad (6.19)$$

$$\kappa - \kappa_\infty = \frac{(f_0 - 1)\{c m (\mu^2 - \nu^2)(1 + f_0) + 2q(m_f - c^2 \tilde{m}_f)\}}{n(m_f^2 + c^2 \tilde{m}_f^2)}, \quad (6.20)$$

while the $U(1)^p$ potentials read

$$v^a = \frac{e^{\phi_\infty} (f_0 - 1)\{c^2 \tilde{m}_f - m_f\} \nu^a - c(1 + f_0)(\mu^a n + 2q \nu^a)}{m_f n_f + c^2 \tilde{n}_f \tilde{m}_f + 2c q (1 - f_0^2)}, \quad (6.21)$$

$$w^a = \frac{e^{-\phi_\infty} (f_0 - 1)\{c^2 \tilde{m}_f - m_f\} (\mu^a n + 2q \nu^a) + c(1 + f_0) \nu^a m^2}{n(m_f^2 + c^2 \tilde{m}_f^2)}, \quad (6.22)$$

where we have chosen $w^a = u^a - \kappa v^a$ instead of u^a which are pure magnetic asymptotically when $\phi_\infty \neq 0, \kappa_\infty \neq 0$.

The solution (6.17), (6.19)–(6.22) constitutes the $SO(2, 2 + p)$ covariantization of the Schwarzschild solution to which it reduces when all parameters $\mu^2, \nu^2, c, \phi_\infty, \kappa_\infty$ are set zero. It contains a maximal number of U -duality parameters compatible with asymptotic conditions and up to $2p$ trivial constants which can be added to v^a, u^a . It has arbitrary values of $2p$ vector charges, ADM mass, NUT parameter, and the scalar charges which are determined in terms of other charges via Eq. (6.15). This latter condition seems to be necessary for the absence of naked singularities (although no general proof has been given so far). Thus, by the uniqueness theorem, which holds for harmonic map into symmetric spaces [24], it is likely to be the most general black hole solution (possibly endowed with a NUT parameter) to $N = 4, D = 4$ supergravity.

VII. DILATON-AXION-REISSNER-NORDSTRÖM BLACK HOLES

Now let us discuss the choice of coordinates for the transformed metric. One reasonable choice is motivated by the Garfinkle-Horowitz-Strominger gauge for a (nondyon) dilatonic black hole [3], in which the curvature singularity sits at some finite radius $r = r_-$:

$$h_{\theta\theta} = f_0 r_0^2 = f r (r - r_-). \quad (7.1)$$

Taking into account that the three-space metric h_{ij} is not changed by transformations, one obtains for r the equation

$$r(r - r_-) = (r_0 - r_0^+)(r_0 - r_0^-), \quad (7.2)$$

where

$$r_0^\pm = \frac{M_0}{m n (1 + c^2)} \{2\alpha - \beta \pm (\beta^2 - 4\alpha\gamma)^{1/2}\}, \quad (7.3)$$

with

$$\alpha = c^2 - 2c q + \mu^2 \nu^2, \quad \beta = (\mu^2 + \nu^2)(1 + c^2), \quad (7.4)$$

$$\gamma = 1 + 2c q + c^2 \mu^2 \nu^2.$$

Clearly

$$r = r_0 - r_0^-, \quad r_- = r_0^+ - r_0^-, \quad (7.5)$$

and hence the four-dimensional metric reads

$$ds^2 = f(dt + 2N \cos\theta d\varphi)^2 - f^{-1} dr^2 - r(r - r_-)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (7.6)$$

where

$$f = 1 - \frac{2M}{r} + \frac{K}{r(r - r_-)}, \quad -K = r_0^+(r_0^- + 2M). \quad (7.7)$$

Here M is a physical mass given by Eq. (6.7). The following relation between the seed and physical masses is useful:

$$M_0 - M = (r_0^+ + r_0^-)/2. \quad (7.8)$$

Unless $K = 0$, g_{00} is not Schwarzschildian, and the metric (7.6) does not coincide with that of [3]. Hence, in the general case the gauge (7.1) has no advantage, and one can choose more familiar curvature coordinates imposing a condition

$$h_{\theta\theta} = f_0 r_0^2 = f \rho^2. \quad (7.9)$$

Then the relation between the old and new coordinates is quadratic, and one obtains

$$r_0 = \frac{1}{2} \{ r_0^+ + r_0^- + (r_-^2 + 4\rho^2)^{1/2} \}. \quad (7.10)$$

In the curvature coordinates the transformed metric is rather similar to the Reissner-Nordström one:

$$ds^2 = \left(1 - \frac{2Mg}{\rho} + \frac{C}{\rho^2} \right) (dt + 2N \cos\theta d\varphi)^2 - \left(1 - \frac{2Mg}{\rho} + \frac{C}{\rho^2} \right)^{-1} d\rho^2 - \rho^2 d\Omega^2, \quad (7.11)$$

with the same mass M and “charge” parameter

$$C = \frac{r_0^-}{2} (r_0^- - 2M_0) + \frac{r_0^+}{2} (r_0^+ - 2M_0). \quad (7.12)$$

It differs, however, from the usual Reissner-Nordström-NUT metric by a “variable mass” factor

$$g = \{ 1 + [r_- / (2\rho)]^2 \}^{1/2}, \quad (7.13)$$

which tends to unity when $\rho \rightarrow \infty$. If $r_- = 0$, the solution is a pure Reissner-Nordström-NUT metric. In this case, in view of Eq. (7.8), $K = C$, and hence Eq. (7.6) gives the same result. In what follows we will refer to the metric (7.11) as dilaton-axion-Reissner-Nordström-NUT (DARN-NUT).

Generally, the equation $g_{00} = 0$ has two solutions, ρ_{\pm} . Taking into account Eq. (7.9), one can realize that smaller root corresponds to $r_0 = 0$, while bigger one to $r_0 = 2M_0$. This means that

$$\rho_-^2 = r_0^- r_0^+, \quad \rho_+^2 = (r_0^- - 2M_0)(r_0^+ - 2M_0), \quad (7.14)$$

or, substituting Eq. (7.3),

$$\rho_{\pm}^2 = \frac{4M_0^2}{mn(1+c^2)} \left(\frac{c^2 \mu^2 \nu^2 + 1 + 2cq}{\mu^2 \nu^2 + c^2 - 2cq} \right). \quad (7.15)$$

Let us consider now the regular DARN black hole, assuming c to obey Eq. (6.10), so that $N = 0$. Parameter r_- entering the above formulas then will be

$$\frac{r_-}{2M} = \frac{\{ (1-c^2)^2 (\mu^2 - \nu^2)^2 + 4c^2 (1 - \mu^2 \nu^2)^2 \}^{1/2}}{(1+c^2)(1 - \mu^2 \nu^2)}, \quad (7.16)$$

while radii of the horizons are

$$\rho_{\pm} = \frac{2Mm^{1/2}n^{1/2}(1-c^2)^{1/2}}{(1+c^2)(1 - \mu^2 \nu^2)} \left(\frac{(1-c^2 \mu^2 \nu^2)^{1/2}}{(\mu^2 \nu^2 - c^2)^{1/2}} \right). \quad (7.17)$$

The solution does not contain naked singularities if $\mu^2 \nu^2 \leq 1$, $c^2 \leq \mu^2 \nu^2$ (which will be assumed below).

The inner horizon is pushed to the singularity when $c^2 = \mu^2 \nu^2$. In this case the solution does not have a NUT parameter if $q = c$. Both relations together imply that in this case the $SO(p)$ vectors μ^a , ν^a are parallel. It can also be checked that $r_0^+ = 0$ and hence $K = 0$, in which case we recover the standard expression for the metric of a “dilaton black hole” [3] with singularity at $r = r_-$,

$$r_- = \frac{2M(\mu^2 + \nu^2)}{1 + \mu^2 \nu^2} = \frac{(Q^a)^2 + (P^a)^2}{M}. \quad (7.18)$$

The electric and magnetic charge vectors are also parallel and in the gauge $b = c$ read

$$Q^a = \frac{\sqrt{2} M_0 (1 + \mu^2) \nu^a}{nm}, \quad P^a = \frac{\sqrt{2} M_0 \mu^a}{m},$$

$$M = \frac{M_0 (1 + \mu^2 \nu^2)}{mn}. \quad (7.19)$$

The dilaton and axion charges are still given by general formulas (6.15) (with $N = 0$) which now read

$$D = \frac{(P^a)^2 - (Q^a)^2}{2M} = \frac{M\{(\mu^2 - \nu^2)(1 - \mu^2 \nu^2) - 4\mu^2 \nu^2\}}{(1 + \mu^2 \nu^2)^2},$$

$$A = -\frac{Q^a P^a}{M} = -\frac{2M(1 + \mu^2)n\mu\nu}{(1 + \mu^2 \nu^2)^2}, \quad (7.20)$$

where μ , ν are norms of the vectors μ^a , ν^a . Note, that in the case of a single ($p = 1$) vector field $q = \mu\nu$ inevitably, and the solution does not contain a NUT parameter if $q = c$, as in the present case. Therefore, the generalization of the “dilaton black hole” solution to the case of multiple vector fields is the DARN solution with parallel $SO(p)$ vectors μ^a , ν^a (or, equivalently, parallel Q^a , P^a). It can be presented in the curvature coordinates as Eq. (7.11), where now

$$C = Mr_- = (Q^a)^2 + (P^a)^2, \quad (7.21)$$

while the radius of the event horizon is

$$\rho_+ = 2M \left(\frac{mn}{1 + \mu^2 \nu^2} \right)^{1/2}. \quad (7.22)$$

Now let us clarify conditions, under which DARN solution becomes *pure* Reissner-Nordström. This happens when $r_- = 0$. From Eq. (7.16) it is clear that one should have $\mu = \nu$, $c = 0$ (thus implying $q = 0$ for vanishing NUT). Therefore the only occurrence of the proper Reissner-Nordström solution in the dilaton-axion gravity with multiple vector fields is the case of the orthogonal $SO(p)$ vectors μ^a , ν^a of equal length. In this case

$$Q^a = \frac{\sqrt{2} M m \nu^a}{1 - \mu^2 \nu^2}, \quad P^a = \frac{\sqrt{2} M n \mu^a}{1 - \mu^2 \nu^2}, \quad (7.23)$$

and the “charge” parameter in Eq. (6.11) assumes its standard form $C = (Q^a)^2 + (P^a)^2$. From Eqs. (6.19), (6.20) it is seen that in this case

$$\phi = \phi_{\infty}, \quad \kappa = \kappa_{\infty} \quad (7.24)$$

(“frozen moduli” [25]). The fact that equal electric and magnetic charges belonging to different $U(1)$ sectors of $N = 4$ supergravity generate the Reissner-Nordström metric was discovered some time ago [5] within a truncated model with no axion. Our results show that this is the *unique* con-

figuration of the $N=4$, $D=4$ supergravity (and its arbitrary- p generalization) ensuring such a property. In particular, if one takes $c=b=0$ (in which case we go back to the intermediate form (6.3) of the solution up to modifications due to an arbitrary asymptotic value z_∞), and $q=0$ to have $N=0$, then

$$r_- = \frac{2M|\mu^2 - \nu^2|}{1 - \mu^2\nu^2} \quad (7.25)$$

and

$$K = \left(\frac{2M}{1 - \mu^2\nu^2} \right)^2 \begin{pmatrix} m^2\nu^2, & \mu^2 > \nu^2 \\ n^2\mu^2, & \nu^2 > \mu^2 \end{pmatrix} \quad (7.26)$$

so the metric is neither Reissner-Nordström, nor dilatonic if $\mu \neq \nu$ and $\mu, \nu \neq 0$. However, it is dilatonic if either all electric, or all magnetic charges are zero (then $K=0$). From this formula one can see that there is another case when $K=0$, namely, when either μ or ν is approaching unity. This is the BPS limit, which is worthy of separate discussion.

Using formulas for physical charges in terms of the parameters one can show that the following identity holds generally

$$M^2 + N^2 + D^2 + A^2 - (Q^a)^2 - (P^a)^2 = M_0^2. \quad (7.27)$$

Thus the BPS limit of the solution corresponds to $M_0 \rightarrow 0$. From Eq. (6.9) it is clear that in the case without a NUT parameter one possibility to achieve this limit (for nonzero M) is to take one of two quantities μ^2 , ν^2 approaching unity and the other still keeping some different value. Then $c \neq 1$, and $r_- \rightarrow 2M$. In the limit one gets the following BPS saturation conditions

$$(Q^a)^2 + (P^a)^2 = 2M^2, \quad D^2 + A^2 = M^2 \quad (7.28)$$

[we choose the gauge $b=c$ in Eq. (6.12)]. From Eq. (7.17) it also follows that both horizon radii shrinks to the singularity. This is the ‘‘dilatonic’’ BPS state which corresponds in $N=4$ supergravity ($p=6$) to 1/2 unbroken supersymmetries [6]. Indeed, absolute values of two central charges in our variables read

$$|z_{1,2}|^2 = \frac{1}{2} \{ (Q^a)^2 + (P^a)^2 \pm \sqrt{(Q^a)^2(P^a)^2 - (Q^a P^a)^2} \}. \quad (7.29)$$

For parallel charge vectors the square root vanishes, so both central charge moduli are equal, and from the saturation condition $|z_{1,2}|=M$ with account for Eq. (7.20) one obtains Eq. (7.28).

If both μ^2 , ν^2 tends to unity simultaneously, then three cases should be distinguished.

(i) μ^a , ν^a are parallel.

In this case $q = \mu\nu$, implying $c = \mu\nu$ to eliminate the NUT parameter, and we come back to the general dilatonic case discussed above.

(ii) μ^a , ν^a are neither parallel, nor orthogonal.

From Eq. (6.10) one finds that then $c \rightarrow 1$ so that

$$\frac{1 - \mu^2\nu^2}{1 - c^2} \rightarrow q, \quad q \neq 0, \quad (7.30)$$

and then $r_- \rightarrow 2M$, $\rho_\pm \rightarrow 0$, hence the limit is also dilatonic. (iii) μ^a , ν^a are orthogonal.

In this case $q=0$, $c=0$, and both horizons merge at

$$\rho_\pm = \rho_{\text{ext}} = \frac{2M\sqrt{mn}}{1 - \mu^2\nu^2}. \quad (7.31)$$

The limit of the right-hand side depends on the curve in the μ, ν plane along which the point $\mu=1$, $\nu=1$ is reached. If one takes the limit along the line $\mu=\nu$, one gets

$$\rho_{\text{ext}} \rightarrow M, \quad (7.32)$$

which corresponds to the extreme Reissner-Nordström black hole. If one goes to the limiting point at some different angle, e.g.,

$$\mu = 1 - \epsilon, \quad \nu = 1 - k\epsilon, \quad \epsilon \rightarrow 0, \quad (7.33)$$

one generally gets an extremal solution with the horizon radius

$$\rho_{\text{ext}}^2 = \frac{4kM^2}{(1+k)^2}. \quad (7.34)$$

In two limiting cases, $k=0$, $k=\infty$ (nondyons) the radius tends to zero, hence the solution is dilatonic. For $k=1$ it is the extremal Reissner-Nordström, for all other values of k this is an extremal limit of the generic solution (7.11). Indeed, the parameter r_- determining the deviation from the Reissner-Nordström geometry now is given by

$$r_- = \frac{2M|1-k|}{1+k}, \quad (7.35)$$

the charge parameter is $C = (Q^a)^2 + (P^a)^2$, while its ratio to the square of mass is

$$\frac{(Q^a)^2 + (P^a)^2}{M^2} = \frac{2(1+k^2)}{(1+k)^2}. \quad (7.36)$$

The dilaton charge is equal to

$$D = \frac{(P^a)^2 - (Q^a)^2}{2M} = \frac{2M^2(k^2 - 1)}{(1+k)^2}, \quad (7.37)$$

while the axion charge is zero, so the force balance condition is satisfied:

$$M^2 + D^2 = (P^a)^2 + (Q^a)^2. \quad (7.38)$$

Furthermore, the electric and magnetic charge vectors have the form

$$Q^a = \frac{\sqrt{2} M \nu^a}{1+k}, \quad P^a = \frac{\sqrt{2} M \mu^a k}{1+k}, \quad (7.39)$$

where μ^a, ν^a are now unit vectors. Hence k can be expressed through the norms of charge vectors $Q = (Q^a Q^a)^{1/2}$, $P = (P^a P^a)^{1/2}$ as

$$k = \frac{P}{Q}. \quad (7.40)$$

From Eq. (7.38) it also follows that

$$P + Q = \sqrt{2}M, \quad (7.41)$$

and then, in view of Eqs. (7.36)–(7.37),

$$P - Q = \sqrt{2}D, \quad (7.42)$$

thus only one BPS bound of $N=4$ supergravity is saturated, signaling that the solution has $N=1$ residual supersymmetry [5,6].

Therefore, in the BPS limit the DARN solution with orthogonal charge vectors has the horizon radius (7.33) taking all values in the interval

$$0 \leq \rho_{\text{ext}} \leq M. \quad (7.43)$$

Zero value is reached for purely electric $k=0$ or purely magnetic $k=\infty$ configurations, while the upper limit corresponds to a symmetric dyon $Q=P$.

To avoid confusion, note that this interpretation is based on our choice of curvature coordinates (like in the standard Reissner-Nordström metric). In [5] it was claimed that *all* BPS dyonic black holes in purely dilaton gravity with two vector fields, possessing electric and magnetic charges in different $U(1)$ sectors, have the same radius of the horizon equal to that of a pure Reissner-Nordström black hole. This may seem to disagree with our conclusion since the particular model of [5] lies within the scope of our consideration. There is no contradiction, however, since in [5] a different coordinate system was used, namely their radial variable (denoted by prime) is related to our r [Eq. (7.1)] as $r' = r - r_-/2$. Taking into account Eq. (7.35), one can easily show that in the BPS limit $r' = M$ independently on k . But the geometrical radius of two spheres in terms of primed variable is equal to $\{r'^2 - (r_-/2)^2\}^{1/2}$, whereas it is ρ^2 in our case. Thus one has the same expression for the area of the horizon surface:

$$\mathcal{A}_H = 4\pi\rho_{\text{ext}}^2 = 8\pi PQ. \quad (7.44)$$

Recall that this expression is valid only in the case of orthogonal charge vectors. If desired, the solution (7.11) may be presented in the form similar to [5]. However, we prefer to use curvature coordinates because of their more transparent geometrical meaning.

VIII. CONCLUSIONS

We have given a concise representation of the three-dimensional U duality for the (generalized) bosonic sector of the stationary $N=4$ supergravity in terms of Kähler parameterization of the target manifold $SO(2,p+2)/[SO(2) \times SO(2,p)]$, where p is a number (chosen arbitrary for generality) of $U(1)$ vector fields. Global $SO(2,p+2)$ symmetry of the three-dimensional theory was described in

terms of holomorphic maps exhibiting the target space isometries. They include Ehlers-Harrison transformations, S -duality, as well as $SO(p)$ rotations, gauge and scale transformations. Selecting transformations which preserve an asymptotic flatness, one obtains a $(2p+4)$ -parametric subset which is suitable to perform a $SO(2,p+2)$ covariantization of any asymptotically flat solution to the vacuum Einstein equations. Such covariantization of the Schwarzschild solution presents the most general static black hole in the dilaton-gravity with multiple vector fields.

Generically DARN black hole solution obtained possesses two horizons. The metric depends on four parameters (in the case without a NUT parameter): the ADM mass, and three $SO(p)$ invariants built out of two $SO(p)$ charge vectors. For parallel charges an internal horizon shrinks to the singularity and the metric coincides with that of the standard dilaton black hole. For orthogonal charge vectors of equal length one has a pure Reissner-Nordström metric. All other charge configurations correspond to generic DARN metric with two nonsingular horizons and a spacelike singularity. The NUT generalization of the solution is also given.

Having now a general nonextremal solution, one can better understand how generic the particular configurations found earlier in the Bogomol'nyi-Gibbons-Hull limit are [5,7,4,6]. One finds that in the BPS limit general DARN solution has a “dilaton” behavior (both horizons shrinking to the singularity), unless charge vectors are orthogonal. Thus the generic BPS static black hole in the $N=4$, $D=4$ supergravity has a vanishing area of the horizon [and $N=2$ residual supersymmetry (SUSY)]. In the case of orthogonal charge vectors two horizons merge at some (generically finite) radius depending on the ratio of norms of the electric and magnetic charge vectors. Nondyon configurations are still dilatonic, while dyons have finite radius of the horizon, taking maximal value in the “symmetric” case.

A few remarks are in order concerning the relationship between our solution generating technique and those, frequently used in the string theory, which are usually based on the presentation of the four-dimensional theory as a dimensional reduction of some higher-dimensional one. First of all, our direct construction of the σ model does not require the theory to be derivable via dimensional reduction. Also, the standard reduction procedure usually gives rise to some matrix representation in terms of real variables which is manifestly invariant under action of the symmetry group, but physical interpretation of transformations often remains obscure. The advantage of the present approach is that it deals with complex potentials, thus reducing the total number of variables by half. It also provides from the very beginning a clear physical identification of the transformations involved. In addition, as we have shown, some discrete symmetries exist which are manifest only in terms of Kählerian variables.

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